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LETTER TO THE EDITOR

Solutions of the boundary Yang–Baxter equation for arbitrary spin

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Abstract

We use boundary quantum group symmetry to obtain recursion formulae which determine nondiagonal solutions of the boundary Yang–Baxter equation (reflection equation) of the XXZ type for any spin j .

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1. Introduction

Solutions $R(u)$ of the Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v) \quad (1.1)$$

play a central role in the study of bulk integrable quantum field theories and solvable lattice models (see, e.g., [1–4]). For simplicity we restrict our attention in this letter to the XXZ case, which is related to the affine Lie algebra $A_1^{(1)}$. Although early investigations focused on the fundamental (spin $\frac{1}{2}$) representation

$$R^{\left(\frac{1}{2}, \frac{1}{2}\right)}(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh u & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh u & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix} \quad (1.2)$$

(where η is the so-called anisotropy parameter), attention soon turned also to higher-dimensional representations. The spin 1 R matrix was obtained [5] by direct solution of the Yang–Baxter equation (1.1). A ‘fusion’ procedure for R matrices was subsequently developed in [6, 7]. However, it was not until a quantum group approach was formulated by Kulish and Reshetikhin [8] that explicit formulae for R matrices of arbitrary spin became available. The key feature of this approach is that it linearizes the problem of finding solutions of the Yang–Baxter equations. This work initiated the study of quantum groups (see, e.g., [9–11]), and seeded important developments in integrable quantum field theory (see, e.g., [12, 13]).

Solutions $K(u)$ of the boundary Yang–Baxter equation

$$R_{12}(u-v)K_1(u)R_{12}(u+v)K_2(v) = K_2(v)R_{12}(u+v)K_1(u)R_{12}(u-v) \quad (1.3)$$

play a corresponding role for quantum integrable models with boundary [14–16]. The (nondiagonal) fundamental representation

$$K^{(\frac{1}{2})}(u) = \begin{pmatrix} \sinh(\xi+u) & \kappa \sinh 2u \\ \kappa \sinh 2u & \sinh(\xi-u) \end{pmatrix} \quad (1.4)$$

(where ξ and κ are boundary parameters) was found [16, 17] by direct solution of (1.3), as was the spin 1 result [18]. A fusion procedure for K matrices was developed in [19–21]. However, the problem of finding explicit formulae for K matrices of arbitrary spin has so far remained unsolved. Some partial results include work on the so-called reflection algebra [22, 23] and on Liouville theory for open strings [24].

We present here some further progress on this problem. Namely, we obtain recursion formulae which determine the matrix elements of $K^{(j)}(u)$ for any spin j . Our approach, generalizing the one used to solve the corresponding bulk problem [8], is based on ‘boundary quantum groups’ [25, 26]. One application of this result is to determine certain coefficients appearing in the Bethe ansatz solution [27] of the open XXZ quantum spin chain with nondiagonal boundary terms at roots of unity.

In section 2 we review the construction of the quantum group generators which commute with the R matrix (1.2) and its higher-spin generalization. In section 3 we first recall [25] the combinations of these quantum group generators which commute with the K matrix (1.4). By demanding that these same combinations of generators also commute with $K^{(j)}(u)$, we obtain a set of linear relations, which we then solve for the matrix elements $K_{mn}^{(j)}(u)$. In section 4 we apply these results to the problem of the open XXZ quantum spin chain with nondiagonal boundary terms at roots of unity. We end with a brief discussion in section 5.

2. The bulk case

In this section, we review the construction [8, 13] of the quantum group generators which commute with the R matrix (1.2) and its higher-spin generalization. To this end, it is convenient to introduce the two-component Faddeev–Zamolodchikov ‘particle-creation operators’

$$A(u) = \begin{pmatrix} A_+(u) \\ A_-(u) \end{pmatrix}$$

in terms of which the R matrix $R^{(\frac{1}{2}, \frac{1}{2})}(u)$ can be defined by

$$A(u_1) \otimes A(u_2) = \check{R}^{(\frac{1}{2}, \frac{1}{2})}(u) A(u_2) \otimes A(u_1) \quad (2.1)$$

where $\check{R}^{(\frac{1}{2}, \frac{1}{2})} = \mathcal{P}R^{(\frac{1}{2}, \frac{1}{2})}$, \mathcal{P} is the permutation matrix and $u = u_1 - u_2$. Associativity of the products in $A(u_1) \otimes A(u_2) \otimes A(u_3)$ then leads [1] to the Yang–Baxter equation (1.1).

Let us assume the following commutation relations of the quantum group generators Q_{\pm} , \bar{Q}_{\pm} and T with the particle-creation operators

$$\begin{aligned} Q_{\pm}A(u) &= q^{\pm\sigma_3} A(u) Q_{\pm} + e^u \sigma_{\mp} A(u) \\ \bar{Q}_{\pm}A(u) &= q^{\mp\sigma_3} A(u) \bar{Q}_{\pm} + e^{-u} \sigma_{\mp} A(u) \\ TA(u) &= A(u)T + \sigma_3 A(u) \end{aligned} \quad (2.2)$$

where

$$q = e^\eta \tag{2.3}$$

and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$. Associativity of the products in $QA(u_1) \otimes A(u_2)|0\rangle$ and invariance of the vacuum $Q|0\rangle = 0$ (where $Q = Q_\pm, T$ or $Q = \bar{Q}_\pm, T$); or, equivalently,

$$\left[\check{R}^{(\frac{1}{2}, \frac{1}{2})}(u), \Delta(Q) \right] = 0 \tag{2.4}$$

(where Δ is the comultiplication) leads to the R matrix (1.2).

This construction generalizes to arbitrary spin $j \in \{\frac{1}{2}, 1, \frac{3}{2}, \dots\}$. We introduce the $(2j + 1)$ -component particle-creation operators $\tilde{A}(u)$, in terms of which the R matrix $R^{(\frac{1}{2}, j)}(u)$ can be defined by

$$A(u_1) \otimes \tilde{A}(u_2) = \check{R}^{(\frac{1}{2}, j)}(u) \tilde{A}(u_2) \otimes A(u_1) \tag{2.5}$$

where $\check{R}^{(\frac{1}{2}, j)} = R^{(\frac{1}{2}, j)} P$, and P is a $2(2j + 1) \times 2(2j + 1)$ matrix which satisfies

$$P(\tilde{M} \otimes N)P^{-1} = N \otimes \tilde{M} \tag{2.6}$$

where \tilde{M} and N are arbitrary $(2j + 1) \times (2j + 1)$ and 2×2 matrices, respectively.

We assume the commutation relations

$$\begin{aligned} Q_\pm \tilde{A}(u) &= q^{\pm 2H} \tilde{A}(u) Q_\pm + e^{u \pm \frac{\eta}{2}} q^{\pm H} E_\mp \tilde{A}(u) \\ \bar{Q}_\pm \tilde{A}(u) &= q^{\mp 2H} \tilde{A}(u) \bar{Q}_\pm + e^{-u \mp \frac{\eta}{2}} q^{\mp H} E_\mp \tilde{A}(u) \\ T \tilde{A}(u) &= \tilde{A}(u) T + 2H \tilde{A}(u) \end{aligned} \tag{2.7}$$

where the matrices H and E_\pm have matrix elements

$$\begin{aligned} (H)_{mn} &= (j + 1 - n)\delta_{m,n} & m, n = 1, 2, \dots, 2j + 1 & & (E_+)_{mn} &= \omega_m \delta_{m, n-1} \\ (E_-)_{mn} &= \omega_n \delta_{m-1, n} & \omega_n &= \sqrt{[n]_q [2j + 1 - n]_q} \end{aligned} \tag{2.8}$$

and

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}. \tag{2.9}$$

These matrices form a $(2j + 1)$ -dimensional representation of the $U_q(su(2))$ algebra

$$[H, E_\pm] = \pm E_\pm \quad [E_+, E_-] = [2H]_q. \tag{2.10}$$

For $j = \frac{1}{2}$, the relations (2.7) reduce to (2.2). Associativity of the products in $QA(u_1) \otimes \tilde{A}(u_2)|0\rangle$ and invariance of the vacuum $Q|0\rangle = 0$ (where $Q = Q_\pm, T$ or $Q = \bar{Q}_\pm, T$) leads to the R matrix [8]

$$R^{(\frac{1}{2}, j)}(u) = \sinh(\eta)(\sigma_+ \otimes E_- + \sigma_- \otimes E_+) + \sinh\left(u + \left(\frac{1}{2} + \sigma_3 \otimes H\right)\eta\right). \tag{2.11}$$

3. The boundary case

Having reviewed the construction of the quantum group generators which commute with the R matrix, we now turn to the boundary case. The matrix $K^{(\frac{1}{2})}(u)$ can be defined by [16]

$$A(u)|0\rangle_B = K^{(\frac{1}{2})}(u)A(-u)|0\rangle_B \tag{3.1}$$

where $|0\rangle_B$ is the vacuum (ground state) in the boundary case. Associativity of the products in $A(u_1) \otimes A(u_2)|0\rangle_B$ then leads [16] to the boundary Yang–Baxter equation (1.3).

Following [25], we consider the combinations of quantum group generators

$$\begin{aligned}\hat{Q} &= \bar{Q}_+ + Q_- - \frac{e^{-\xi}}{2\kappa \sinh \eta} q^{-T} \\ \hat{Q}' &= \bar{Q}_- + Q_+ + \frac{e^{\xi}}{2\kappa \sinh \eta} q^T\end{aligned}\quad (3.2)$$

which generate the boundary quantum group. Indeed, associativity of the products in $\hat{Q}A(u)|0\rangle_B$ and $\hat{Q}'A(u)|0\rangle_B$, together with invariance of the vacuum $Q_{\pm}|0\rangle_B = \bar{Q}_{\pm}|0\rangle_B = T|0\rangle_B = 0$, imply (using the commutation relations (2.2)) the K matrix (1.4).

The spin j matrix $K^{(j)}(u)$ can similarly be defined by

$$\tilde{A}(u)|0\rangle_B = K^{(j)}(u)\tilde{A}(-u)|0\rangle_B. \quad (3.3)$$

Associativity of the products in $\hat{Q}\tilde{A}(u)|0\rangle_B$ and $\hat{Q}'\tilde{A}(u)|0\rangle_B$ and invariance of the vacuum imply (using the commutation relations (2.7))

$$\begin{aligned}&\left(e^{-u-\frac{\eta}{2}} q^{-H} E_- + e^{u+\frac{\eta}{2}} q^{-H} E_+ - \frac{e^{-\xi}}{2\kappa \sinh \eta} q^{-2H} \right) K^{(j)}(u) \\ &= K^{(j)}(u) \left(e^{u-\frac{\eta}{2}} q^{-H} E_- + e^{-u+\frac{\eta}{2}} q^{-H} E_+ - \frac{e^{-\xi}}{2\kappa \sinh \eta} q^{-2H} \right)\end{aligned}\quad (3.4)$$

and

$$\begin{aligned}&\left(e^{-u-\frac{\eta}{2}} q^H E_+ + e^{u+\frac{\eta}{2}} q^H E_- + \frac{e^{\xi}}{2\kappa \sinh \eta} q^{2H} \right) K^{(j)}(u) \\ &= K^{(j)}(u) \left(e^{u-\frac{\eta}{2}} q^H E_+ + e^{-u+\frac{\eta}{2}} q^H E_- + \frac{e^{\xi}}{2\kappa \sinh \eta} q^{2H} \right)\end{aligned}\quad (3.5)$$

respectively. Making use of the explicit expressions (2.8) for the matrix elements of H and E_{\pm} , we obtain the relations

$$\begin{aligned}&e^{-u-\eta(j+\frac{3}{2}-m)} \omega_{m-1} K_{m-1,n}^{(j)}(u) + e^{u-\eta(j+\frac{1}{2}-m)} \omega_m K_{m+1,n}^{(j)}(u) - \frac{e^{-\xi}}{2\kappa \sinh \eta} e^{-2\eta(j+1-m)} K_{mn}^{(j)}(u) \\ &= e^{u-\eta(j+\frac{1}{2}-n)} \omega_n K_{m,n+1}^{(j)}(u) + e^{-u-\eta(j+\frac{3}{2}-n)} \omega_{n-1} K_{m,n-1}^{(j)}(u) \\ &\quad - \frac{e^{-\xi}}{2\kappa \sinh \eta} e^{-2\eta(j+1-n)} K_{mn}^{(j)}(u)\end{aligned}\quad (3.6)$$

and

$$\begin{aligned}&e^{-u+\eta(j+\frac{1}{2}-m)} \omega_m K_{m+1,n}^{(j)}(u) + e^{u+\eta(j+\frac{3}{2}-m)} \omega_{m-1} K_{m-1,n}^{(j)}(u) + \frac{e^{\xi}}{2\kappa \sinh \eta} e^{2\eta(j+1-m)} K_{mn}^{(j)}(u) \\ &= e^{u+\eta(j+\frac{3}{2}-n)} \omega_{n-1} K_{m,n-1}^{(j)}(u) + e^{-u+\eta(j+\frac{1}{2}-n)} \omega_n K_{m,n+1}^{(j)}(u) \\ &\quad + \frac{e^{\xi}}{2\kappa \sinh \eta} e^{2\eta(j+1-n)} K_{mn}^{(j)}(u)\end{aligned}\quad (3.7)$$

where $K_{mn}^{(j)}(u)$ denotes the (m, n) matrix element of $K^{(j)}(u)$. It is understood that these matrix elements vanish for index values outside the range $[1, 2j+1]$.

The relations (3.6) and (3.7) determine the matrix $K^{(j)}(u)$, up to an overall unitarization factor which does not concern us here. Indeed, we find that this matrix is symmetric

$K_{mn}^{(j)}(u) = K_{nm}^{(j)}(u)$, and³

$$\begin{aligned}
 K_{mn}^{(j)}(u) &= \kappa^{n-m} \sqrt{\prod_{l=0}^{n-m-1} \frac{\sinh((2j-m+1-l)\eta)}{\sinh((n-1-l)\eta)}} \prod_{l=0}^{m-2} \frac{\sinh((n-1-l)\eta)}{\sinh((m-1-l)\eta)} \\
 &\times \prod_{l=0}^{n-m-1} \sinh(2u-l\eta) \prod_{l=0}^{2j-n} \sinh\left(\xi+u+\left(l-j+\frac{1}{2}\right)\eta\right) \\
 &\times \prod_{l=0}^{m-2} \sinh\left(\xi-u-\left(l-j+\frac{1}{2}\right)\eta\right) J_{mn}^{(j)}(u), \\
 &m, n = 1, 2, \dots, 2j+1 \quad n \geq m
 \end{aligned} \tag{3.8}$$

where the quantities $J_{mn}^{(j)}(u)$ are given by

$$J_{mn}^{(j)}(u) = \sum_{k=0}^{\lfloor \frac{2j+m-n}{2} \rfloor} \kappa^{2k} J_{mn}^{(j,k)}(u) \quad J_{mn}^{(j,0)}(u) = 1. \tag{3.9}$$

Finally, let us describe the quantities $J_{mn}^{(j,k)}(u)$ for $k \geq 1$: for $m = 1$, they are given by

$$J_{1,n}^{(j,k)}(u) = \sum_{l_1=0}^{2j-1-n} \sum_{l_2=l_1+2}^{2j-1-n} \dots \sum_{l_k=l_{k-1}+2}^{2j-1-n} F_{l_1}(u, j; n) F_{l_2}(u, j; n) \dots F_{l_k}(u, j; n) \tag{3.10}$$

where

$$F_l(u, j; n) = \frac{\sinh(2u - (n+l)\eta) \sinh((2j-n-l)\eta)}{\sinh\left(\xi+u+\left(j+\frac{1}{2}-n-l\right)\eta\right) \sinh\left(\xi+u+\left(j-\frac{1}{2}-n-l\right)\eta\right)}. \tag{3.11}$$

For $m \geq 2$, these quantities are determined (in terms of the quantities with $m = 1$ (3.10)) by the recursion relations

$$\begin{aligned}
 J_{mn}^{(j,k)}(u) &= a_{mn}^{(j)}(u) J_{m-1,n-1}^{(j,k)}(u) + b_{mn}^{(j)}(u) J_{m-1,n}^{(j,k)}(u) + c_{mn}^{(j)}(u) J_{m-2,n}^{(j,k-1)}(u) \\
 &m = 2, 3, \dots, 2j+1
 \end{aligned} \tag{3.12}$$

where

$$\begin{aligned}
 a_{mn}^{(j)}(u) &= \frac{\sinh\left(\xi+u+\left(j-n+\frac{3}{2}\right)\eta\right) \sinh(2u+\eta)}{\sinh((n-m+1)\eta) \sinh\left(\xi-u+\left(j-m+\frac{3}{2}\right)\eta\right)} \\
 b_{mn}^{(j)}(u) &= -\frac{\sinh\left(\xi+u+\left(j-m+\frac{5}{2}\right)\eta\right) \sinh(2u-(n-m)\eta)}{\sinh((n-m+1)\eta) \sinh\left(\xi-u+\left(j-m+\frac{3}{2}\right)\eta\right)} \\
 c_{mn}^{(j)}(u) &= -\frac{\sinh((m-2)\eta) \sinh((2j-m+3)\eta)}{\sinh((n-m+2)\eta) \sinh^2((n-m+1)\eta)} \\
 &\times \frac{\sinh(2u+(n-m+2)\eta) \sinh(2u-(n-m+1)\eta) \sinh(2u-(n-m)\eta)}{\sinh\left(\xi-u+\left(j-m+\frac{5}{2}\right)\eta\right) \sinh\left(\xi-u+\left(j-m+\frac{3}{2}\right)\eta\right)}.
 \end{aligned} \tag{3.13}$$

The recursion relation (3.12) is satisfied for $k = 0$ by virtue of the identity

$$1 = a_{mn}^{(j)}(u) + b_{mn}^{(j)}(u). \tag{3.14}$$

The recursion relations (3.6) and (3.7), and the expressions (3.8)–(3.13) for the matrix elements of $K^{(j)}(u)$ constitute the main results of this letter. We have explicitly verified for values of

³ Due to the presence of the square root (which originates from the factors ω_n (2.8)), we expect that for $m \neq n$ this result is strictly valid only for η real. For η imaginary, some phase factors may appear.

spin up to $j = 2$ that these results agree with those obtained by fusion [20, 21, 27], up to a shift of the spectral parameter and an overall factor.

It is easy to see from equation (3.8) that $K_{mn}^{(j)}(0)$ vanishes for $m \neq n$; and, in fact, is proportional to δ_{mn} , as follows from also equations (3.12), (3.13). Furthermore, the dependence of $K_{mn}^{(j)}(u)$ on the boundary parameter κ is given by $\sim \kappa^{|n-m|}$, plus terms that are of higher order in κ . Hence, for $\kappa = 0$, $K_{mn}^{(j)}(u)$ is diagonal, and is entirely given by equation (3.8)—no recursion relation is then needed, since the quantities $J_{mn}^{(j,k)}(u)$ do not depend on κ . We also remark that the symmetry $K_{nm}^{(j)}(u) = K_{mn}^{(j)}(u)$ follows from the symmetry of the equations (3.6), (3.7) under transposition of K and simultaneous relabelling $n \leftrightarrow m$. As already observed in [16], one can break this symmetry and introduce a third parameter α into the K matrix by performing a change of basis $\tilde{A}(u) \mapsto e^{i\alpha H} \tilde{A}(u)$. While this leaves the R matrix unchanged, it transforms the entries of the K matrix as $K_{mn} \mapsto e^{i\alpha(m-n)} K_{mn}$.

It is tempting to conjecture that there exist generalizations of the formulae (3.10), (3.11) which are valid not just for $m = 1$, but for all values of m . Indeed, we have found that an expression of the form

$$J_{mn}^{(j,k)}(u) = \sum_{l_1=1-m}^{2j-1-n} \sum_{l_2=l_1+2}^{2j-1-n} \dots \sum_{l_k=l_{k-1}+2}^{2j-1-n} F_{l_1}(u, j; m, n) F_{l_2}(u, j; m, n) \dots F_{l_k}(u, j; m, n) \quad (3.15)$$

holds for values of m up to at least $m = 4$. However, we have not yet succeeded in finding general formulae for the corresponding functions $F_l(u, j; m, n)$.

4. An application

One immediate application of our result is to determine certain coefficients appearing in the Bethe ansatz solution of the open XXZ quantum spin chain with nondiagonal boundary terms, defined by the Hamiltonian [15, 17]

$$\mathcal{H} = \frac{1}{2} \left\{ \sum_{n=1}^{N-1} (\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \cosh \eta \sigma_n^z \sigma_{n+1}^z) + \sinh \eta \left(\coth \xi_- \sigma_1^z + \frac{2\kappa_-}{\sinh \xi_-} \sigma_1^x - \coth \xi_+ \sigma_N^z - \frac{2\kappa_+}{\sinh \xi_+} \sigma_N^x \right) \right\}. \quad (4.1)$$

We recall [27] that for bulk anisotropy value

$$\eta = \frac{i\pi}{p+1}, \quad p = 1, 2, \dots \quad (4.2)$$

(and hence $q = e^\eta$ is a root of unity, satisfying $q^{p+1} = -1$), the spin- $\frac{p+1}{2}$ transfer matrix can be expressed in terms of a lower-spin transfer matrix, resulting in the truncation of the fusion hierarchy. In order to obtain this crucial ‘truncation identity’ (which in turn leads to a functional relation for the fundamental transfer matrix, and then finally to a set of Bethe ansatz-like equations for the transfer-matrix eigenvalues), one needs some knowledge of the matrix $K^{(j)}(u)$ with $j = \frac{p+1}{2}$. In particular, for the (1, 1) matrix element, it was conjectured in [27] that

$$K_{11}^{(\frac{p+1}{2})} \text{previous}(u) \propto n(u; \xi, \kappa) = \sinh((p+1)(\xi+u)) + \sum_{k=1}^{[\frac{p+1}{2}]} c_{p,k} \kappa^{2k} \sinh((p+1)u + (p+1-2k)\xi) \quad (4.3)$$

where $c_{p,k}$ are some unknown coefficients. These coefficients were explicitly computed in [27] for values of p up to $p = 5$, and they were found to be consistent with the formulae

$$\begin{aligned} c_{p,1} &= p + 1 \\ c_{p,2} &= \frac{1}{2}p(p - 1) - 1. \end{aligned} \tag{4.4}$$

We have designated by ‘previous’ the K -matrix appearing in [27], in order not to confuse it with the K -matrix used here, from which it differs by a shift of spectral parameter and an overall factor,

$$K_{\text{previous}}^{(j)}(u) \propto K^{(j)}\left(u + \left(j - \frac{1}{2}\right)\eta\right). \tag{4.5}$$

Using our results (3.8)–(3.11) for $m = n = 1$, we obtain

$$\begin{aligned} K_{11}^{(\frac{p+1}{2})}(\text{previous})(u) &\propto \sinh((p + 1)(\xi + u)) \\ &\times \left\{ 1 + \sum_{k=1}^{\lfloor \frac{p+1}{2} \rfloor} \kappa^{2k} \sum_{l_1=0}^{p-1} \sum_{l_2=l_1+2}^{p-1} \dots \sum_{l_k=l_{k-1}+2}^{p-1} f_{l_1}(u; p) f_{l_2}(u; p) \dots f_{l_k}(u; p) \right\} \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} f_l(u; p) &= F_l\left(u + \left(j - \frac{1}{2}\right)\eta, j; 1\right) \Big|_{\eta=\frac{i\pi}{p+1}, j=\frac{p+1}{2}} \\ &= -\frac{\sinh(2u - (l + 2)\eta) \sinh((l + 1)\eta)}{\sinh(\xi + u - (l + 2)\eta) \sinh(\xi + u - (l + 1)\eta)} \Big|_{\eta=\frac{i\pi}{p+1}}. \end{aligned} \tag{4.7}$$

Using the identity

$$\begin{aligned} \sum_{l_1=0}^{p-1} \sum_{l_2=l_1+2}^{p-1} \dots \sum_{l_k=l_{k-1}+2}^{p-1} f_{l_1}(u; p) f_{l_2}(u; p) \dots f_{l_k}(u; p) \\ = \left(\frac{p+1}{k!} \prod_{l=0}^{k-2} (p - k - l)\right) \frac{\sinh((p + 1)u + (p + 1 - 2k)\xi)}{\sinh((p + 1)(\xi + u))} \end{aligned} \tag{4.8}$$

and comparing (4.6) and (4.8) with (4.3), we conclude that the coefficients $c_{p,k}$ are given by

$$c_{p,k} = \frac{(p + 1)}{k!} \prod_{l=0}^{k-2} (p - k - l) = \frac{p + 1}{k} \binom{p - k}{k - 1}. \tag{4.9}$$

This result is evidently consistent with (4.4).

With these coefficients in hand, the Bethe ansatz equations can be written down from [27] for all the η values (4.2). In particular, it becomes possible to study the $p \rightarrow \infty$ limit, for which $c_{p,k} \sim \frac{p^k}{k!}$.

5. Discussion

We have found expressions (3.8)–(3.13) for the matrix elements of $K^{(j)}(u)$, for arbitrary spin j . Since the K matrix depends on two boundary parameters ξ and κ as well as the bulk anisotropy parameter η , one should *a priori* expect the expression for the K matrix to be more complicated than that of the R matrix (2.11). Our result certainly bears this out. Nevertheless, we expect that it may be possible to simplify our result, perhaps along the lines of (3.15). Indeed, a better understanding of the boundary quantum group symmetry may lead to a better choice of variables with which to express the K matrix.

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